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Maps on quantum states preserving the Jensen–Shannon divergence

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Abstract
We show that any bijective map on the space of all quantum states which preserves the Jensen–Shannon divergence is induced by a unitary or antiunitary operator.

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1. Introduction and statement of the result

As is well known, distinguishability measures between quantum states play a central role in quantum theory, especially in the areas of quantum computation and quantum information. Many of these measures have origins in classical probability theory, statistics or information theory. In fact, most of them were originally defined for probability distributions and later extended to quantum states. This has happened, among others, to the Kolmogorov (or variational) distance and the Kullback–Leibler divergence. Their quantum analogs are the trace distance (more precisely, its $\frac{1}{2}$-multiple) and the relative entropy, respectively. However, there are distinguishability measures between quantum states which have no classical probabilistic origin. As an example we mention the so-called Bures metric. This concept is closely related to the notion of Uhlmann's fidelity which is an extension of the fundamental concept of transition probability from the case of pure states to the case of mixed states.

Transformations on the space of quantum states which preserve a given physically relevant relation or quantity can be considered as a kind of symmetries of the underlying quantum system. For example, the maps on pure states which preserve the above-mentioned transition probability are usually called quantum mechanical symmetry transformations. Wigner's celebrated theorem (see, e.g., [2, section 2.2]) concerning the structure of those maps states

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that every quantum mechanical symmetry transformation is induced by a unitary or antiunitary operator on the Hilbert space that corresponds to the underlying quantum system. Motivated by Wigner's result, in our paper [8] we determined the structure of all maps on the whole space of quantum states which preserve the Bures metric or the trace distance. Moreover, in the recent work [7] the first author obtained a similar description for the transformations on the space of states which leave the relative entropy invariant. It is well known (and, in fact, easy to see) that any transformation implemented by a unitary or antiunitary operator has all these invariance or preserver properties. The joint content of our results in [7, 8] is that there are no other kinds of transformations with any of these properties different from that induced by unitary or antiunitary operators. For further results on preservers of quantum structures we refer to the second chapter of the volume [6]. Roughly speaking, by a preserver we mean a transformation on a mathematical structure which preserves 'something' which is relevant for the underlying structure (this 'something' can be a given quantity, relation, operation, a collection of distinguished elements, etc).

Recently, in their paper [5] Majtey, Lamberti and Prato have introduced the concept of Jensen–Shannon divergence for quantum states. This is a modification of the notion of relative entropy and also has an origin in classical information theory. In fact, Rao [1, chapter 5] and Lin [4] defined the Jensen–Shannon divergence for probability distributions as a symmetrized version of the Kullback-Leibler divergence. That new divergence has many advantages: it is always well defined and bounded, and its square root gives a true metric on the probability distribution space. Besides classical information theory, the concept has proved to be useful in other areas, for example, in some problems of statistical physics. The quantum Jensen–Shannon divergence is a direct generalization of that concept for the case of quantum states. Its various properties and the advantages of its use in quantum theory as a new distinguishability measure between states were treated in the papers [3, 5]. It was pointed out there that this divergence inherits several important, characteristic properties of the relative entropy. It is invariant under unitary transformations, non-increasing under CP (completely positive) maps and non-increasing when taking traces over parts of a composite system. Moreover, it is jointly convex, it has a restricted additivity property and satisfies a kind of Donald's identity. However, in contrast to the relative entropy, it is always well defined and bounded by 1. It is symmetric and, in fact, conjectured to be the square of a true metric on the state space. The quantum Jensen–Shannon divergence is closely related to other quantum distances. It can be applied to approximate the Wooters distance, its generalization defined by Braunstein and Caves, and the Bures metric. It can be used to define a good entanglement measure, and it can be interpreted as the upper bound for the accessible quantum information.

The aim of the present paper is to show that the symmetries of the space of states with respect to the Jensen–Shannon divergence, i.e., the transformations which leave it invariant have the same structure as in the case of the previously mentioned distinguishability measures. Namely, every such a map is induced by a unitary or antiunitary operator on the underlying Hilbert space.

Let us begin with the notation and the necessary definitions. In what follows, H is a (complex) Hilbert space with dimension $2 \leqslant \dim H < \infty$. A state on H is a positive (semi-definite) operator with unit trace. The set of all states on H is denoted by $\mathcal{S}(H)$. The elements of $\mathcal{S}(H)$ are also called density operators. In what follows, 'log' denotes the binary logarithm and 'ln' stands for the natural logarithm.

Let $\rho, \sigma \in \mathcal{S}(H)$ be states. The von Neumann entropy of ρ is the non-negative real number

$$S(\rho) = -\mathrm{tr}\,\rho \log \rho.$$

The relative entropy between ρ and σ is defined by

$$S(\rho\|\sigma) = \operatorname{tr}\rho(\log\rho - \log\sigma)$$

(recall that this quantity can be infinite). The basic concept of this paper is the Jensen–Shannon divergence. It is defined between ρ and σ by the formula

$$D_{JS}(\rho\|\sigma) = \frac{1}{2}\left[S\left(\rho\left\|\frac{\rho+\sigma}{2}\right.\right) + S\left(\sigma\left\|\frac{\rho+\sigma}{2}\right.\right)\right].$$

In terms of the von Neumann entropy $D_{JS}(\rho\|\sigma)$ can be expressed as

$$D_{JS}(\rho\|\sigma) = S\left(\frac{\rho+\sigma}{2}\right) - \frac{1}{2}S(\rho) - \frac{1}{2}S(\sigma). \tag{1}$$

As mentioned above, unlike the relative entropy, the Jensen–Shannon divergence is symmetric in its two variables and it is always a non-negative real number (it cannot be infinite).

Among the fundamental properties of D_{JS} discussed in [5], it was mentioned in the first place that the Jensen–Shannon divergence is invariant under unitary (and also antiunitary) transformations. Our result states that in fact there are no other kinds of bijective transformations on the space of states having that invariance property. The precise statement reads as follows.

Theorem. *Let $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ be a bijective map which preserves the Jensen–Shannon divergence, i.e., which satisfies*

$$D_{JS}(\phi(\rho)\|\phi(\sigma)) = D_{JS}(\rho\|\sigma)$$

for every $\rho, \sigma \in \mathcal{S}(H)$. Then there is either a unitary or antiunitary operator U on H such that ϕ is of the form

$$\phi(\rho) = U\rho U^* \qquad (\rho \in \mathcal{S}(H)).$$

2. Proof

Before presenting the proof, we note that in the rest of the paper we follow the usual functional analytical notation and conventions concerning Hilbert spaces and their operators. Namely, the inner product $\langle .,.\rangle$ on H is linear in its first variable and conjugate-linear in the second. This is the reverse of what is common in physics. By a projection we mean a self-adjoint idempotent. The rank of an operator is the dimension of its range, tr denotes the usual trace functional on operators. If $x, y \in H$ are arbitrary vectors, then $x \otimes y$ denotes the operator (of rank at most 1) defined by

$$(x \otimes y)(z) = \langle z, y\rangle x \qquad (z \in H).$$

Clearly, the rank-1 projections (i.e., pure states) are exactly the operators of the form $x \otimes x$ with some unit vector $x \in H$. A small remark should be added here concerning the use of the sign $\otimes$ of tensor product. Usually $x \otimes y$ has to be considered as an element (elementary tensor) of the tensor product space $H \otimes H$. But it is well known that the (complete) tensor product $H \otimes H$ can be identified (via a unitary transformation) with the space of all Hilbert–Schmidt operators on H. Under this identification the elementary tensor $x \otimes y$ corresponds exactly to the operator we have defined above. This is the reason why we use the same symbol for that operator.

We learn from [5] that

$$D_{JS}(\rho\|\sigma) \leqslant 1$$

holds for arbitrary $\rho, \sigma \in \mathcal{S}(H)$ and that

$$D_{JS}(\rho\|\sigma) = 1 \iff \text{supp}\,\rho \perp \text{supp}\,\sigma.$$

Here supp ρ denotes the support of ρ, which is the orthogonal complement of its kernel (which is equal to the range of the operator ρ) and $\perp$ stands for the relation of orthogonality between subsets of the Hilbert space H. In what follows we shall usually use the shorter notation $\rho \perp \sigma$ instead of supp $\rho \perp$ supp σ and speak about orthogonality between states.

After these preliminaries we can present the proof of the theorem.

Proof. The short outline of the proof is the following: firstly, we prove that the transformation ϕ maps rank-1 projections to rank-1 projections and preserves the transition probability. We next apply Wigner's theorem and obtain the form of ϕ on the set of all rank-1 projections. In the remaining and most essential part of the proof we verify that the form is valid for all density operators.

To work out this plan, we first observe that as the orthogonality relation $\rho \perp \sigma$ between the elements $\rho, \sigma \in \mathcal{S}(H)$ can be characterized by the equality $D_{JS}(\rho\|\sigma) = 1$, it follows that ϕ preserves orthogonality in both directions. This means that for arbitrary $\rho, \sigma \in \mathcal{S}(H)$ we have $\rho \perp \sigma$ if and only if $\phi(\rho) \perp \phi(\sigma)$.

For any collection $\mathcal{C}$ of states we write $\mathcal{C}^\perp$ to denote the set of all elements of $\mathcal{S}(H)$ which are orthogonal to every element of $\mathcal{C}$. It is easy to see that $\rho \in \mathcal{S}(H)$ is a rank-1 projection if and only if $\{\rho\}^{\perp\perp} = \{\rho\}$. As ϕ is a bijection on $\mathcal{S}(H)$ which preserves the orthogonality in both directions, using the above characterization we easily obtain that $\rho \in \mathcal{S}(H)$ is a rank-1 projection (i.e., a pure state) if and only if $\phi(\rho)$ is. This means that the restriction of ϕ on the set $\mathcal{P}_1(H) \subset \mathcal{S}(H)$ of all rank-1 projections maps $\mathcal{P}_1(H)$ onto itself. Our next aim is to show that this restriction as a transformation on pure states preserves the transition probability.

Pick two rank-1 projections p, q on H. It follows from (1) that

$$D_{JS}(p\|q) = S\left(\frac{p+q}{2}\right)$$

and this quantity can easily be computed in the following way. First define a real-valued function on the closed unit interval [0, 1] by

$$f(t) = \begin{cases} -\left(\frac{1+t}{2}\log\frac{1+t}{2} + \frac{1-t}{2}\log\frac{1-t}{2}\right), & 0 \leqslant t < 1; \\ 0, & t = 1. \end{cases}$$

We assert that

$$D_{JS}(p\|q) = S\left(\frac{p+q}{2}\right) = f(\sqrt{\text{tr}\, pq}) \tag{2}$$

(cf (18) in [3]). In fact, in order to calculate $S\left(\frac{p+q}{2}\right)$ we have to find the eigenvalues of the positive operator $p+q$. Suppose $p \neq q$ (otherwise the case is trivial) and pick unit vectors x, y from the ranges of p and q, respectively. We have $p = x \otimes x$ and $q = y \otimes y$. The matrix representation of the restriction of $p+q$ onto the subspace generated by x, y in the (not necessarily orthonormal) basis $\{x, y\}$ is

$$\begin{pmatrix} 1 & \langle y, x\rangle \\ \langle x, y\rangle & 1 \end{pmatrix}.$$

It follows immediately that the non-zero eigenvalues of $p+q$ are $1 \pm |\langle x, y\rangle|$. Moreover, as

$$pq = x \otimes x \cdot y \otimes y = \langle y, x\rangle \cdot x \otimes y$$

and $\operatorname{tr} x \otimes y = \langle x, y\rangle$, we obtain the equality

$$\sqrt{\operatorname{tr} pq} = |\langle x, y\rangle|.$$

Now (2) follows.

Elementary differential calculus gives that the above-defined f is a strictly decreasing function on $[0, 1]$ and hence it is injective. As ϕ preserves the Jensen–Shannon divergence between rank-1 projections, it follows from (2) that

$$f(\sqrt{\operatorname{tr} \phi(p)\phi(q)}) = D_{JS}(\phi(p)\|\phi(p)) = D_{JS}(p\|q) = f(\sqrt{\operatorname{tr} pq}).$$

Using the injectivity of f we infer that

$$\operatorname{tr} \phi(p)\phi(q) = \operatorname{tr} pq$$

holds for all $p, q \in \mathcal{P}_1(H)$. This means that the restriction of ϕ on $\mathcal{P}_1(H)$ preserves the transition probability and hence it is a quantum mechanical symmetry transformation. By Wigner's theorem it follows that there exists a unitary or antiunitary operator U on H such that

$$\phi(p) = UpU^* \qquad (p \in \mathcal{P}_1(H)).$$

Define a new transformation $\psi : \mathcal{S}(H) \to \mathcal{S}(H)$ by

$$\psi(\rho) = U^*\phi(\rho)U \qquad (\rho \in \mathcal{S}(H)). \tag{3}$$

We have already mentioned above that the Jensen–Shannon divergence is invariant under unitary–antiunitary transformations. Therefore, $\psi : \mathcal{S}(H) \to \mathcal{S}(H)$ is a bijective map which preserves the Jensen–Shannon divergence and has the additional property that $\psi(p) = p$ holds for every rank-1 projection p. In the remaining part of the proof we shall show that $\psi(\rho) = \rho$ holds for every $\rho \in \mathcal{S}(H)$, too.

Pick a state $\rho \in \mathcal{S}(H)$ and denote $\rho' = \psi(\rho)$. For every $p \in \mathcal{P}_1(H)$ we have

$$D_{JS}(p\|\rho) = D_{JS}(\psi(p)\|\psi(\rho)) = D_{JS}(p\|\rho'). \tag{4}$$

We compute

$$\begin{aligned}
D_{JS}(p\|\rho) &= S\left(\frac{p+\rho}{2}\right) - \frac{1}{2}S(\rho)\\
&= -\operatorname{tr}\left(\frac{p+\rho}{2}\right)\log\left(\frac{p+\rho}{2}\right) + \frac{1}{2}\operatorname{tr}\rho\log\rho\\
&= -\frac{1}{2}\operatorname{tr}[(p+\rho)(\log(p+\rho) - I)] + \frac{1}{2}\operatorname{tr}\rho\log\rho\\
&= -\frac{1}{2}\operatorname{tr}[(p+\rho)\log(p+\rho)] + 1 + \frac{1}{2}\operatorname{tr}\rho\log\rho\\
&= -\frac{1}{2}\operatorname{tr}[(p+\rho)\log(p+\rho) - \rho\log\rho] + 1.
\end{aligned}$$

Applying (4) we obtain that the equality

$$\operatorname{tr}[(p+\rho)\log(p+\rho) - \rho\log\rho] = \operatorname{tr}[(p+\rho')\log(p+\rho') - \rho'\log\rho'] \tag{5}$$

holds for every rank-1 projection $p \in \mathcal{P}_1(H)$.

Now pick an arbitrary rank-1 projection p and an arbitrary self-adjoint operator S. Insert the rank-1 projections on the curve $t \mapsto \mathrm{e}^{\mathrm{i}tS} p\, \mathrm{e}^{-\mathrm{i}tS}$, $t \in \mathbb{R}$ into the equality (5). We obtain that

$$\begin{aligned}
&\operatorname{tr}[(\mathrm{e}^{\mathrm{i}tS} p\, \mathrm{e}^{-\mathrm{i}tS} + \rho)\log(\mathrm{e}^{\mathrm{i}tS} p\, \mathrm{e}^{-\mathrm{i}tS} + \rho) - \rho\log\rho]\\
&\quad = \operatorname{tr}[(\mathrm{e}^{\mathrm{i}tS} p\, \mathrm{e}^{-\mathrm{i}tS} + \rho')\log(\mathrm{e}^{\mathrm{i}tS} p\, \mathrm{e}^{-\mathrm{i}tS} + \rho') - \rho'\log\rho'].
\end{aligned}$$

As the function log is a constant multiple of the function ln, we trivially have the modified equality

$$\begin{aligned}\mathrm{tr}[(\mathrm{e}^{\mathrm{i}tS}p\,\mathrm{e}^{-\mathrm{i}tS}+\rho)\ln(\mathrm{e}^{\mathrm{i}tS}p\,\mathrm{e}^{-\mathrm{i}tS}+\rho)-\rho\ln\rho]\\ =\mathrm{tr}[(\mathrm{e}^{\mathrm{i}tS}p\,\mathrm{e}^{-\mathrm{i}tS}+\rho')\ln(\mathrm{e}^{\mathrm{i}tS}p\,\mathrm{e}^{-\mathrm{i}tS}+\rho')-\rho'\ln\rho']\end{aligned}\tag{6}$$

that holds for every $t\in\mathbb{R}$. A small but important observation should be made here. Recall that every bijective map on $\mathcal{S}(H)$ which preserves the Jensen–Shannon divergence also preserves the orthogonality between the elements of $\mathcal{S}(H)$. As ψ acts like the identity on $\mathcal{P}_1(H)$, it follows that a rank-1 projection is orthogonal to ρ if and only if it is orthogonal to ρ'. This gives us that the supports of ρ and ρ' are necessarily the same. In what follows, we restrict our considerations to this particular subspace denoted by H_0. Suppose that the previously chosen rank-1 projection p projects onto a one-dimensional subspace of H_0 and that the self-adjoint operator S also acts on H_0 (i.e., it is zero on the orthogonal complement of H_0). In that way we can assume that in (6) only invertible operators appear in the arguments of the logarithms.

We now consider both sides of (6) as functions of the real variable t and differentiate them at the point $t=0$. First we have to check that

$$t\longmapsto\mathrm{tr}[(\mathrm{e}^{\mathrm{i}tS}p\,\mathrm{e}^{-\mathrm{i}tS}+\rho)\ln(\mathrm{e}^{\mathrm{i}tS}p\,\mathrm{e}^{-\mathrm{i}tS}+\rho)-\rho\ln\rho]\tag{7}$$

is a differentiable function. Although this is a real-valued function of one real variable, it is in fact the composition of several more complicated functions with operator values and/or operator variables. First, observe that the trace functional is linear and hence it is differentiable. The differentiability of the one-parameter family $t\mapsto\mathrm{e}^{\mathrm{i}tS}$ is well known and its derivative is $t\mapsto\mathrm{e}^{\mathrm{i}tS}\cdot\mathrm{i}S$. There is no problem with differentiating a fixed operator multiple of an operator-valued differentiable function of one real variable or the pointwise product of two such functions. Thus, the only delicate point concerns the logarithm function which has to be considered here as a transformation defined on the invertible positive operators and taking values in the space of self-adjoint operators. In (7) this transformation is composed from the right by the operator-valued function $t\mapsto\mathrm{e}^{\mathrm{i}tS}p\,\mathrm{e}^{-\mathrm{i}tS}+\rho$ of the real variable t. From section 4 in [9] we learn the following: the operator logarithm function is Fréchet-differentiable and its derivative at a point T_0 is given by the linear transformation

$$\left.\frac{\mathrm{d}\ln T}{\mathrm{d}T}\right|_{T=T_0}:X\longmapsto\int_0^\infty(T_0+\lambda I)^{-1}X(T_0+\lambda I)^{-1}\,\mathrm{d}\lambda.\tag{8}$$

Using the above information and applying some rules of differentiation (e.g., the Leibniz rule, chain rule, etc), it is now not difficult to check that the function in (7) is differentiable and its derivative at the point $t=0$ is given by

$$\begin{aligned}\mathrm{tr}\Big[&((\mathrm{i}S)p+p(-\mathrm{i}S))\ln(p+\rho)\\ &+(p+\rho)\int_0^\infty(p+\rho+\lambda I)^{-1}((\mathrm{i}S)p+p(-\mathrm{i}S))(p+\rho+\lambda I)^{-1}\,\mathrm{d}\lambda\Big].\end{aligned}$$

Dividing this quantity by the imaginary number i we obtain

$$\mathrm{tr}[(Sp-pS)\ln(p+\rho)]+\mathrm{tr}\left[(p+\rho)\int_0^\infty(p+\rho+\lambda I)^{-1}(Sp-pS)(p+\rho+\lambda I)^{-1}\,\mathrm{d}\lambda\right].\tag{9}$$

We would like to write the second trace value above in a more simple form. To do this, we use the fact that because of the continuity and the linearity of tr, the order of taking the trace and taking the integral can be interchanged and also apply the characteristic identity $\operatorname{tr} AB = \operatorname{tr} BA$ which holds for all operators A, B on H. We then compute

$$\begin{aligned}
&\operatorname{tr}\left[(p+\rho)\int_0^\infty (p+\rho+\lambda I)^{-1}(Sp-pS)(p+\rho+\lambda I)^{-1}\,\mathrm{d}\lambda\right]\\
&\qquad = \operatorname{tr}\left[\int_0^\infty (p+\rho)(p+\rho+\lambda I)^{-1}(Sp-pS)(p+\rho+\lambda I)^{-1}\,\mathrm{d}\lambda\right]\\
&\qquad = \int_0^\infty \operatorname{tr}[(p+\rho)(p+\rho+\lambda I)^{-1}(Sp-pS)(p+\rho+\lambda I)^{-1}]\,\mathrm{d}\lambda\\
&\qquad = \int_0^\infty \operatorname{tr}[(p+\rho+\lambda I)^{-1}(p+\rho)(p+\rho+\lambda I)^{-1}(Sp-pS)]\,\mathrm{d}\lambda\\
&\qquad = \operatorname{tr}\left[\int_0^\infty (p+\rho+\lambda I)^{-1}(p+\rho)(p+\rho+\lambda I)^{-1}(Sp-pS)\,\mathrm{d}\lambda\right]\\
&\qquad = \operatorname{tr}\left[\left(\int_0^\infty (p+\rho+\lambda I)^{-1}(p+\rho)(p+\rho+\lambda I)^{-1}\,\mathrm{d}\lambda\right)(Sp-pS)\right]\\
&\qquad = \operatorname{tr}\left[(p+\rho)\left(\int_0^\infty (p+\rho+\lambda I)^{-2}\,\mathrm{d}\lambda\right)(Sp-pS)\right].
\end{aligned}$$

The value of the integral $\int_0^\infty (p+\rho+\lambda I)^{-2}\,\mathrm{d}\lambda$ can be determined in several ways. One of the possibilities is to use another result from the paper [9]. Clearly, by the formula (8) we have

$$\int_0^\infty (p+\rho+\lambda I)^{-2}\,\mathrm{d}\lambda = \left(\frac{\mathrm{d}\ln T}{\mathrm{d}T}\bigg|_{T=p+\rho}\right)(I).$$

As I commutes with $p+\rho$, we can use [9, proposition 2.2] to see that

$$\left(\frac{\mathrm{d}\ln T}{\mathrm{d}T}\bigg|_{T=p+\rho}\right)(I) = (\ln'(p+\rho))\cdot I = (p+\rho)^{-1}.$$

Therefore, we obtain for the second trace value in (9) that

$$\begin{aligned}
&\operatorname{tr}\left[(p+\rho)\int_0^\infty (p+\rho+\lambda I)^{-1}(Sp-pS)(p+\rho+\lambda I)^{-1}\,\mathrm{d}\lambda\right]\\
&\qquad = \operatorname{tr}\left[(p+\rho)\left(\int_0^\infty (p+\rho+\lambda I)^{-2}\,\mathrm{d}\lambda\right)(Sp-pS)\right] = \operatorname{tr}[Sp-pS] = 0.
\end{aligned}$$

Consequently, the derivative of the function in (7) at $t=0$ equals

$$\mathrm{i}\operatorname{tr}[(Sp-pS)\ln(p+\rho)].$$

It follows that differentiating both sides of the equation (6) with respect to the parameter t at the point $t_0=0$, we obtain that

$$\mathrm{i}\operatorname{tr}[(Sp-pS)\ln(p+\rho)] = \mathrm{i}\operatorname{tr}[(Sp-pS)\ln(p+\rho')].$$

Remember that this holds for every rank-1 projection p and self-adjoint operator S.

Pick any unit vector x and set $p = x\otimes x$. Denote $T_x = \ln(p+\rho) - \ln(p+\rho')$. (Observe that ρ and hence ρ' as well have been fixed above, but as x varies, the operator T_x also varies.) It follows that

$$\operatorname{tr}[(Sp-pS)T_x] = 0$$

holds for every self-adjoint operator S. Choose an arbitrary unit vector y which is orthogonal to x and set $S = y \otimes x + x \otimes y$. Then we have

$$0 = \mathrm{tr}[(Sp - pS)T_x] = \mathrm{tr}[(y \otimes x - x \otimes y)T_x] = \langle y, T_x x\rangle - \langle x, T_x y\rangle.$$

Thus we obtain that

$$\langle y, T_x x\rangle = \langle x, T_x y\rangle$$

holds for every unit vector y which is orthogonal to x. Inserting iy in the place of y (this is also a unit vector which is orthogonal to x) we see that $\langle(\mathrm{i}y), T_x x\rangle = \langle x, T_x(\mathrm{i}y)\rangle$ implying

$$\langle y, T_x x\rangle = -\langle x, T_x y\rangle.$$

Therefore, we deduce that $\langle y, T_x x\rangle = 0$ holds for every unit vector y which is orthogonal to x. This gives us that $T_x x$ is a scalar multiple of x for every unit vector x.

Let x be a normalized eigenvector of ρ and set $p = x \otimes x$ as above. Clearly x is an eigenvector of $p + \rho$. This implies that x is an eigenvector of $\ln(p + \rho)$. But as x is an eigenvector of T_x, by the definition of T_x we obtain that x is an eigenvector of $\ln(p+\rho')$. This further implies that x is an eigenvector of $p + \rho'$ and hence also an eigenvector of ρ'. Clearly, in a similar manner one can verify that every normalized eigenvector of ρ' is an eigenvector of ρ. So, we conclude that the eigenvectors of ρ and ρ' are the same and this implies that these two operators can be diagonalized in the same orthonormal basis. Consider the matrix representations of ρ and ρ' in that basis

$$\rho = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \qquad \rho' = \begin{pmatrix} \lambda'_1 & & \\ & \ddots & \\ & & \lambda'_n \end{pmatrix}$$

and let p be the rank-1 projection onto the subspace generated by the ith basis vector. Computing both sides of (5) we easily obtain that

$$(\lambda_i + 1)\ln(\lambda_i + 1) - \lambda_i \ln\lambda_i = (\lambda'_i + 1)\ln(\lambda'_i + 1) - \lambda'_i \ln\lambda'_i.$$

We assert that this implies $\lambda_i = \lambda'_i$. To see this, consider the real function $g(t) = (t + 1)\ln(t + 1) - t\ln t$ defined for every $0 < t < 1$. By differentiation, one can check that g is strictly increasing and hence injective. Therefore, we obtain that $\lambda_i = \lambda'_i$. As this equality holds for every $1 \leqslant \mathrm{i} \leqslant n$, we get the desired equality $\rho = \rho' = \psi(\rho)$.

To sum up, we have proved above that the transformation ψ defined in (3) is the identity on $\mathcal{S}(H)$. This means that $U^*\phi(\rho)U = \rho$ which clearly implies

$$\phi(\rho) = U\rho U^*$$

for every $\rho \in \mathcal{S}(H)$. The proof of the theorem is complete. □

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